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## CHARACTERIZATIONS OF NORMAL APPROXIMATE SPECTRA

M. FUJII and C.-S. LIN

1. Introduction. In [4], one of the authors introduced a normal spectrum of an operator in a  $C^*$ -algebra  $A$ . An operator  $T$  is a normal topological divisor of zero if there is a sequence  $\{A_n\}$  of operators in  $A$  such as  $\|A_n\| = 1$ ,

$$\|TA_n\| \longrightarrow 0 \quad \text{and} \quad \|T^*A_n\| \longrightarrow 0.$$

The set  $\sigma_n(T)$  of all  $z$ 's such that  $T - z$  is a normal topological divisor of zero is called here the normal spectrum of  $T$ .

Such a kind of normality of spectra of operators is introduced by several authors [2,3,5] independently about ten years ago, cf. also [1]. A scalar  $z$  is called a normal approximate propervalue of  $T$  if one of the following conditions is satisfied:

(i) There is a sequence  $\{x_n\}$  of unit vectors such as

$$\|(T - z)x_n\| \longrightarrow 0 \quad \text{and} \quad \|(T - z)^*x_n\| \longrightarrow 0.$$

(ii) There is no  $s > 0$  such as

$$(T - z)^*(T - z) + (T - z)(T - z)^* \geq s.$$

(iii) There is a character  $\phi$  of the  $C^*$ -algebra  $C^*(T)$  generated by  $T$  and the identity such as  $z = \phi(T)$ .

All normal approximate propervalues of  $T$  form a compact set  $\pi_n(T)$ , which is called the normal approximate spectrum of  $T$  by [1]. By (ii) and (iii), it is clear that  $\pi_n(T)$  is purely algebraic, which is determined within  $C^*(T)$ . So we have the following problems: (1) Is the normal

spectrum purely algebraic ? (2) Are there any relations between  $\sigma_n(T)$  and  $\pi_n(T)$  ? (They are not discussed in [4].)

In this note, we shall give a solution to the above problems as follows: They are just the same (considering a C\*-algebra acts faithfully on a Hilbert space). As an application, one can give a C\*-algebraic proof to the reciprocity stated above in (iii).

2. Normal spectra. Now we shall give a C\*-algebraic characterization of the normal approximate spectrum.

Theorem. The normal spectrum is nothing but the normal approximate spectrum:  $\sigma_n(T) = \pi_n(T)$ .

Proof. First note that  $0 \in \sigma_n(T)$  if and only if there is a sequence  $\{A_n\}$  of positive operators in  $A$  such that  $\|A_n\| = 1$ ,

$$TA_n \longrightarrow 0 \text{ and } T^*A_n \longrightarrow 0.$$

Suppose that  $0 \in \pi_n(T)$ . Since  $T^*T + TT^*$  is not invertible by (ii), there is a sequence  $\{A_n\}$  of positive operators in  $A$  such that  $\|A_n\| = 1$ ,

$$(T^*T + TT^*)A_n \longrightarrow 0.$$

Since  $A_n(T^*T + TT^*)A_n \longrightarrow 0$ , we have

$$A_n T^* T A_n \longrightarrow 0 \text{ and } A_n T T^* A_n \longrightarrow 0,$$

or equivalently

$$TA_n \longrightarrow 0 \text{ and } T^*A_n \longrightarrow 0.$$

Conversely, assume that  $0 \notin \pi_n(T)$ , i.e.,  $T^*T + TT^* \geq s$  for some  $s > 0$ . For any  $B \geq 0$  with  $\|B\| = 1$ , since

$$BT^*TB + BTT^*B \geq sB^2,$$

it follows that

$$\|TB\|^2 + \|T^*B\|^2 \geq s,$$

which implies that  $0 \notin \sigma_n(T)$ .

3. Applications. In this section, we shall give another proofs to the following characterizations of normal approximate propervalue.

Corollary 1. For  $T$  in a unital  $C^*$ -algebra  $A$ , a scalar  $z$  belongs to  $\pi_n(T)$  if and only if the left ideal generated by  $T$  and  $T^*$  is proper in  $A$ , i.e.,

$$A(T - z) + A(T - z)^* \neq A.$$

Proof. Suppose that  $0 \in \pi_n(T) = \sigma_n(T)$ . Then there is a sequence  $\{B_n\}$  in  $A$  such that  $\|B_n\| = 1$ ,  $TB_n \rightarrow 0$  and  $T^*B_n \rightarrow 0$ . If

there exist  $A$  and  $B$  in  $A$  such that  $AT + BT^* = 1$ , then

$$1 = \|B_n\| = \|ATB_n + BT^*B_n\| \leq \|A\|\|TB_n\| + \|B\|\|T^*B_n\| \rightarrow 0.$$

This is a contradiction.

Conversely, if  $0 \notin \pi_n(T)$ , then  $T^*T + TT^*$  is invertible. Since  $(AT^*)T + (AT)T^* = 1$  for some  $A$  in  $A$ , it follows that  $AT + AT^* = A$ .

Finally we shall give a simple proof to the following reciprocity;

Corollary 2. For  $T$  in a unital  $C^*$ -algebra, a scalar  $z$  belongs to  $\pi_n(T)$  if and only if there is a character  $\varphi$  on  $C^*(T)$  such as  $\varphi(T) = z$ .

Proof. If  $0 \in \pi_n(T)$ , then there exists  $\{B_n\}$  in  $C^*(T)$  such that  $TB_n \rightarrow 0$ ,  $T^*B_n \rightarrow 0$  and  $\|B_n\| = 1$ . Since  $\|B_n\| = 1$ , there is a state  $f_n$  on  $C^*(T)$  such that  $f_n(B_n^*B_n) = 1$ . So we define a state  $\varphi$  by the composition of a Banach limit  $\text{Lim}$  on  $l^\infty$ ;

$$\varphi(A) = \text{Lim } f_n(B_n^*AB_n) \quad \text{for } A \in C^*(T).$$

Since  $\varphi(p(T, T^*)) = p(0, 0)$  for any non-commutative polynomial  $p$  on  $T$  and  $T^*$ , a state  $\varphi$  is a character with  $\varphi(T) = 0$ .

Conversely, if there is a character  $\varphi$  such that  $\varphi(T) = 0$ , then  $\varphi(C^*(T)T + C^*(T)T^*) = 0$ . Therefore we have  $C^*(T)T + C^*(T)T^* \neq C^*(T)$ , which implies that  $0 \in \pi_n(T)$  by Corollary 1.

Remark. In the final part of the above, Corollary 1 is not necessary, cf. [1; I, Theorem 1]: If  $\varphi(T) = 0$  for some character  $\varphi$ , then

$$\varphi(T^*T + TT^*) = 0.$$

Since  $\varphi(1) = 1$ , it is impossible that there is  $s > 0$  such as  $T^*T + TT^* \geq s$ .

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